MATH 2040B 2022-23

Midterm 1

Q1. (a). Yes.

For $(a_1, a_2, a_3), (a_1', a_2', a_3') \in S_1$.

$$(a_1+a_1') - 4(a_2+a_2') - (a_3+a_3') = (a_1+4a_2-a_3) + (a_1'+4a_2'-a_3') = 0$$

we have $(a_1+a_1', a_2+a_2', a_3+a_3') \in S_1$.

For $a \in \mathbb{R}$. $(a_1, a_2, a_3) \in S_1$.

$$a \cdot a_1 - 4 a \cdot a_2 - a \cdot a_3 = a(a_1 - 4a_2 - a_3) = 0$$ (5')

we have $a \cdot (a_1, a_2, a_3) \in S_1$.

Hence S_1 is a vector subspace.

(b) No.

$(1,0,1) \in S_2$. $(1,0,-1) \in S_2$ but $(2,0,0) = (1,0,1) + (1,0,-1) \notin S_2$. (5')

(c). Yes.

For $f_1, f_2 \in S_3$, $f_1+f_2 \in S_3$ as

$$(f_1+f_2)(-x) = f_1(-x) + f_2(-x) = -f_1(x) - f_2(x) = -(f_1+f_2)(x).$$ (5')

For $a \in \mathbb{R}$ $f \in S_3$, $af \in S_3$ as

$$(af)(-x) = af(-x) = -af(x) = -(af)(x).$$

(d). No.

$$\begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} \in S_4 \text{ but } \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix} + \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} = id \notin S_4.$$ (5')

Q2. (a) $\dim S_1 = 2$.

Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$, $\varphi(a_1, a_2, a_3) = a_1 - 4a_2 - a_3$. (5')

This is a linear map by above, and surjective as $\varphi(a,0,0) = a$ for all $a \in \mathbb{R}$.

$\ker \varphi = S_1$. By rank-nullity theorem, $\dim S_1 = 3 - 1 = 2$.

(c). $\dim S_3 = \lfloor \frac{n+1}{2} \rfloor = \begin{cases} \frac{n+1}{2} & n \text{ odd} \\ \frac{n}{2} & n \text{ even} \end{cases}$ (5')

Claim: $\{x, x^3, \cdots, x^r\}$ is a basis for S_3, where r is greatest odd integer $\leq n$.

Pf. Since they all have odd degree, $(-x)^{2k+1} = (-1)^{2k+1} x^{2k+1} = -x^{2k+1}$, so they are in S_3.

They are linearly independent as elements in $\mathbb{P}_n(\mathbb{R})$, so linearly independent.

It remains to show they span S_3.

Take $f(x) = a_0 + a_1 x + \cdots + a_n x^n$.

$-f(x) = f(-x)$ gives

$$-a_0 - a_1 x - \cdots - a_n x^n = a_0 - a_1 x + \cdots + (-1)^n a_n x^n.$$

so

$$a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{2k} x^{2k} = 0$$ where $2k$ is greatest even integer $\le n$.

Taking $x = 0, 1, \cdots, k$ gives system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & (2^2)^2 & \cdots & (2^2)^k \\ \vdots & & & & \\ 1 & k^2 & (k^2)^2 & \cdots & (k^2)^k \end{pmatrix} \begin{pmatrix} a_0 \\ a_2 \\ a_4 \\ \vdots \\ a_{2k} \end{pmatrix} = 0.$$

By Vandermonde determinant formula, $\det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & & 1 \\ 1 & 2^2 & \cdots & & (2^2)^k \\ \vdots & & & & \\ 1 & k^2 & \cdots & & (k^2)^k \end{pmatrix} \neq 0.$

Hence $a_0 = a_2 = \cdots = a_{2k} = 0$.

Hence $f(x) \in \mathrm{span}\{x, x^3, \cdots, x^r\}$.

As a result, $\{x, x^3, \cdots, x^r\}$ is a basis for S_3. Its size is $\lfloor \frac{n+1}{2} \rfloor$.

Q2. (a) Yes.

For $A = (a_{ij})$, $B = (b_{ij}) \in M_{3\times 3}(\mathbb{R})$, $a \in \mathbb{R}$. (5')

$$tr(A+B) = \sum_{i=1}^{3} (a_{ii} + b_{ii}) = \sum_{i=1}^{3} a_{ii} + \sum_{i=1}^{3} b_{ii} = tr(A) + tr(B).$$

$$tr(aA) = \sum_{i=1}^{3} a a_{ii} = a \sum_{i=1}^{3} a_{ii} = a\, tr(A).$$

Hence tr is a linear map.

Claim: $tr^{-1}(0)$ has basis $\left\{ \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{pmatrix} \right\} \cup \{ E_{ij} : (i,j)\text{-entry is } 1, \text{ other entries are } 0 \mid i \neq j \}$

proof. tr is surjective, $tr\begin{pmatrix} a&0&0\\0&0&0\\0&0&0 \end{pmatrix} = a$ for all $a \in \mathbb{R}$, so by rank-nullity theorem, $\dim tr^{-1}(0) = 8$. As we only have 8 vectors, it suffices to show they are linearly independent.

Assume $\sum_{i \neq j} a_{ij} E_{ij} + b \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix} + c \begin{pmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{pmatrix} = 0$, $a_{ij}, b, c \in \mathbb{R}$. (10')

By looking at entries (i,j), $i \neq j$, we conclude $a_{ij} = 0$ for all $i \neq j$.

By looking at entry $(1,1)$ and $(3,3)$, $b = c = 0$.

Hence they are linearly independent, and hence a basis.

(b) No. $\det\left(\begin{pmatrix} 1&0\\0&0 \end{pmatrix} + \begin{pmatrix} 0&0\\0&1 \end{pmatrix} \right) = 1 \neq 0 = \det \begin{pmatrix} 1&0\\0&0 \end{pmatrix} + \det \begin{pmatrix} 0&0\\0&1 \end{pmatrix}$.

(5')

Q3. (a). Let $v = \sum_{j=1}^{n} a_j v_j$ be the <u>unique</u> way of writing v into linear combination of basis.

Define $T(v) = \sum_{j=1}^{n} a_i T(v_j) = \sum_{j=1}^{n} a_i (v_j - 2v_{j-1})$. It is well-defined by uniqueness.

Let $v' = \sum_{j=1}^{n} a_j' v_j$. $a \in \mathbb{R}$. (10')

Let $v+v' = \sum_{j=1}^{n} b_j v_j = \sum_{j=1}^{n} a_j v_j + \sum_{j=1}^{n} a_j' v_j$

As the way of writing $v+v'$ into linear combination of basis is unique,

$b_j = a_j + a_j'$ for all $j = 1, \cdots, n$.

Hence $T(v+v') = T(v) + T(v')$

Let $av = \sum_{j=1}^{n} c_j v_j = a \sum_{j=1}^{n} a_j v_j$.

By uniqueness again, $c_j = a a_j$ for all $j = 1, \cdots, n$.

Hence $T(av) = aT(v)$.

Hence T is a linear map.

(b). As $T(v_j) = 0v_1 + \cdots + 0v_{j-2} + (-2)v_{j-1} + v_j + 0v_{j+1} + \cdots + 0 \cdot v_n$. for all j.

$$[T]_\beta = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & & & 1 & -2 \\ 0 & 0 & \cdots & & & & 1 \end{pmatrix}$$ (10')

(c). Yes. As $\det [T]_\beta = 1 \neq 0$, T is invertible. (10')

Q4. (a). $T_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. $T_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$ be a basis of $\mathbb{R}^2$

Let α be the standard basis.

$[id]_\beta^\alpha = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$. $[id^{-1}]_\alpha^\beta = \left([id]_\beta^\alpha\right)^{-1} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}^{-1} = \frac{1}{10}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$. $[T]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (10')

Hence $[T]_\alpha^\alpha = [Id]_\beta^\alpha [T]_\beta^\beta [id^{-1}]_\alpha^\beta = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$

(b). As $0x + 3y - z = 0$, $\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ is a normal vector to the plane $3y - z = 0$. $T_2 \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$.

Also, $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is on the plane, so $T_2 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$. $T_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right\}$ be a basis of $\mathbb{R}^3$ and α be the standard basis. (10')

Then $[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. $[id]_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$. $[id^{-1}]_\alpha^\beta = \left([id]_\beta^\alpha\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & \frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{pmatrix}$

Hence $[T]_\alpha^\alpha = [id]_\beta^\alpha [T]_\beta^\beta [id^{-1}]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & \frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$